Eggleston meets Mycielski, measure case

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Eggleston Theorem

For every conull set $F \subseteq [0,1]^2$ there are a perfect set $P \subseteq [0,1]$ and conull $B \subseteq [0,1]$ such that $P \times B \subseteq F$.

H. G. Eggleston, Two measure properties of Cartesian product sets, The Quarterly Journal of Mathematics 5 (1954), 108–115.

Mycielski theorem

For every conull set $F \subseteq [0,1]^2$ there exists a perfect set $P \subseteq [0,1]$ such that $P \times P \subseteq F \cup \Delta$, where $\Delta = \{(x,x) : x \in [0,1]\}$.

J. Mycielski, Algebraic independence and measure, Fundamenta Mathematicae 61 (1967) 165-–169.

Definition

A tree $T \subseteq 2^{<\omega}$ is

- perfect if $(\forall \sigma \in T)(\exists \tau \supseteq \sigma)(\tau^\frown 0, \tau^\frown 1 \in T)$;
- a Silver tree if T is perfect and

$$(\exists x \in 2^\omega)(\exists A \in [\omega]^\omega)(\forall \sigma \in T)(\forall n \in \mathrm{dom}(\sigma))$$
$$(n \notin A \to \sigma(n) = x(n));$$

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Definition

$A \subseteq 2^\omega$ is a small set if there is a partition $\mathcal{A}$ of ω into finite sets and a collection $(J_a)_{a \in \mathcal{A}}$ such that $J_a \subseteq 2^a$, $\sum_{a \in \mathcal{A}} \frac{|J_a|}{2^{|a|}} < \infty$ and

$$A = \{x \in 2^\omega : (\exists^\infty a \in \mathcal{A})(x \restriction a \in J_a)\}.$$

Theorem about Silver trees, Mycielski case

There exist a small set $A \subseteq 2^\omega \times 2^\omega$ such that $(A \cap [T] \times [T]) \setminus \Delta \neq \emptyset$ for any Silver tree $T \subseteq 2^{<\omega}$.

Theorem about Silver trees, Mycielski case

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Proof

Let $\{I_n\}_{n\in\omega}$ be a partition of ω such that $|I_n| \geq n$.
Define

$$J_{n,m} = \begin{cases} \emptyset & \text{if } \; n \neq m \\ \{(x,x) : \; x \in 2^{I_n}\} & \text{if } \; n = m \end{cases}$$

$$A = \{(x,y) \in 2^\omega \times 2^\omega : \; (\exists^\infty n \in \omega)(x \restriction I_n = y \restriction I_n)\}$$

is a small set.

□

Theorem about Silver trees, Eggleston case

For every conull set $F \subseteq (2^\omega \times 2^\omega)$ there are a Silver tree $T \subseteq 2^{<\omega}$ and F_σ conull set $H \subseteq 2^\omega$ such that $[T] \times H \subseteq F$.

Key lemma

Let $\varepsilon > 0$, $F \subseteq 2^\omega$ closed, $\sigma \in 2^{<\omega}$, $H \subseteq 2^\omega$ a union of basic clopen sets of size $2^{-|\sigma|}$, satisfying $F \subseteq [\sigma] \times H$ and $\lambda(F) > (1-\varepsilon^2)\lambda([\sigma] \times H)$.
Then there exists $X \subseteq [\sigma]$ satisfying $\lambda(X) > (1-\varepsilon)\lambda([\sigma])$ such that for each $x \in X$

$$(\star) \quad (\forall \delta > 0)(\exists N \in \omega)(\forall n \geq N)(\exists S_n \subseteq 2^n)$$
$$(\lambda(\bigcup_{\tau \in S_n}[\tau]) > (1-\varepsilon)\lambda(H) \wedge$$
$$\wedge\ (\forall \tau \in S_n)(\lambda(F \cap [x \upharpoonright n] \times [\tau]) > (1-\delta)2^{-2n})).$$

Definition

A tree $T \subseteq 2^{<\omega}$ is a Spinas tree if

$$(\forall \tau \in T)(\exists N \in \omega)(\forall n \geq N)(\forall i \in 2)$$
$$(\exists \tau' \in T \cap 2^{n+1})(\tau \subseteq \tau' \wedge \tau'(n) = i).$$

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Theorem about Spinas trees

For every set $F \subseteq (2^\omega \times 2^\omega)$ there are a Spinas tree $T \subseteq 2^{<\omega}$ and F_σ conull set $B \subseteq 2^\omega$ such that $[T] \times B \subseteq F$. Moreover, T contains a Silver tree.

Definition

A tree $T \subseteq 2^{<\omega}$ is uniformly perfect if it is perfect and

$$(\forall \sigma, \tau \in T)((|\sigma| = |\tau|) \to (\sigma^\frown 0, \sigma^\frown 1 \in T \to \tau^\frown 0, \tau^\frown 1 \in T)).$$

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Theorem where Mycielski meets Eggleston

For every conull set $F \subseteq (2^\omega \times 2^\omega)$ there are a uniformly perfect tree $T \subseteq 2^{<\omega}$ and F_σ conull set $B \subseteq 2^\omega$ such that $[T] \subseteq B$ and $[T] \times B \subseteq F \setminus \Delta$.

Thank you for your attention!

https://prac.im.pwr.edu.pl/~twowlc

References

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- Sz. Żeberski, Nonstandard proofs of Eggleston like theorems, Proceedings of the Ninth Topological Symposium (2001), 353–357.